Blacktown Boys' High School

2018

HSC Trial Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 7 – 11)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Assessor: X. Chirgwin

Student Name: ____

Teacher Name: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2018 Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1–10

Use the multiple choice answer sheet provided on page 13 for Questions 1–10.

---

1 Which of the following is an expression for $\int \sin^2 8x\, dx$?

A. $\frac{1}{2}x + \frac{1}{32}\sin 16x + C$

B. $\frac{1}{2}x - \frac{1}{32}\sin 16x + C$

C. $\frac{1}{2}x + \frac{1}{16}\sin 8x + C$

D. $\frac{1}{2}x - \frac{1}{16}\sin 8x + C$

2 Given that $f(x) = \frac{5}{x+2} + 3$.

The equations of the asymptotes of the graph of the inverse function $f^{-1}(x)$ are:

A. $x = -2$ and $y = 3$

B. $x = -2$ and $y = -3$

C. $x = 3$ and $y = -2$

D. $x = -3$ and $y = -2$

3 Which of the following is the range of the function $y = \frac{1}{2}\cos^{-1} x - \frac{\pi}{2}$?

A. $0 \le y \le \frac{\pi}{2}$

B. $0 \le y \le \pi$

C. $-\frac{\pi}{2} \le y \le 0$

D. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

4 How many arrangements in a row of the letters of the word AMPLITUDE are possible if all the vowels are together in any order?

A. $6! \times 4!$

B. $5! \times 4!$

C. $9!$

D. $4!$

5 In what ratio does the point $R(8, -1)$ divides the interval AB, where A and B are $(-4, 3)$ and $(5, 0)$ respectively?

A. $-1 : 4$

B. $-4 : 1$

C. $4 : 1$

D. $1 : 4$

6 The displacement x of a particle at time t is given by $x = 6\sin 5t + 8\cos 5t$. What is the greatest speed of the particle?

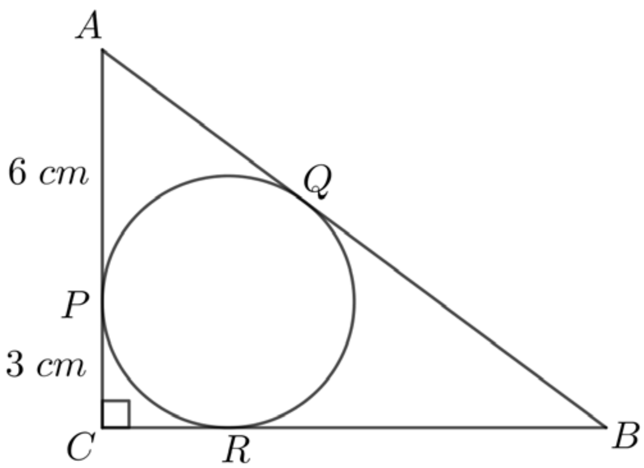
A. 70

B. 50

C. 30

D. 10

7 In the diagram, AC is a tangent to the circle at the point P, AB is a tangent to the circle at the point Q, and BC is a tangent to the circle at the point R. Find the exact length of BC if $CP = 3\ cm$ and $AP = 6\ cm$.



A. $6\ cm$

B. $9\ cm$

C. $12\ cm$

D. $15\ cm$

8 Which expression is equal to $\dfrac{{}^{n}P_{r}}{{}^{n}C_{r}}$?

A. $\dfrac{n!}{(n-r)!}$

B. $\dfrac{n!}{r!}$

C. $\dfrac{1}{r!}$

D. $r!$

9 Let $|p| \leq 1$, what is the general solution of $\cos 2x = p$?

A. $x = n\pi \pm \dfrac{\cos^{-1} p}{2}$, n is an integer

B. $x = n\pi \pm \cos^{-1}\dfrac{p}{2}$, n is an integer

C. $x = 2n\pi \pm \cos^{-1}\dfrac{p}{2}$, n is an integer

D. $x = \dfrac{n\pi + (-1)^{n}\cos^{-1} p}{2}$, n is an integer

10 The velocity $v\ ms^{-1}$ of a particle moving in a straight line is governed by the equation $v = x - 3$, where x is its displacement. Initially, the particle was at $x = 7\ cm$. What is the displacement function of this particle?

A. $x = 7 + e^{t}$

B. $x = 7e^{t}$

C. $x = 3 + e^{t}$

D. $x = 3 + 4e^{t}$

End of Section I

Section II

60 Marks
Attempt Questions 11–14
Answer each question on a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Solve $\dfrac{4x}{x+5} < 2$. **2**

b) Find the acute angle between the lines $x - 5y = 0$ and $2x + y - 3 = 0$. Leave your answer to the nearest minute. **2**

c) Find $\dfrac{d}{dx}(x\sin^{-1}x + \sin^{-1}x)$ **2**

d) The variable point $(5\cos\theta, 5\sin\theta)$ lies on a curve. Find the Cartesian equation of this curve. **2**

e) i) Prove that $\cot\left(\dfrac{\theta}{2}\right) = \dfrac{\sin\theta}{1-\cos\theta}$. **2**

ii) Hence find the exact value of $\cot\left(\dfrac{\theta}{2}\right)$ given that $\sin\theta = \dfrac{5}{6}$ and $\dfrac{\pi}{2} < \theta < \pi$. **2**

f) Use the substitution $u = 2e^x$ to show that $\displaystyle\int_{\ln\frac{1}{2}}^{\ln\frac{\sqrt{3}}{2}} \frac{e^x}{1+4e^{2x}}\,dx = \frac{\pi}{24}$. **3**

End of Questions 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Consider the function $f(x) = \log_e(2x - 1) + x^3 + 1$.

i) Show that a root exists between $x = 0.6$ and $x = 0.7$. **1**

ii) Use one application of Newton's method, starting at $x = 0.6$, to find another approximation of this root correct to 3 significant figures. **2**

b) The equation $3x^3 - 7x - 2 = 0$ has roots $\alpha,\ \beta$ and γ. Find the value of:

i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

c) A Mathematics club consists of 20 members, of which there are 11 men and 9 women. A committee of four people is to be chosen randomly.

i) How many committees can there be if there is to be equal numbers of men and women on the committee? **1**

ii) How many committees can there be if, regardless of their gender, the four-member committee must contain the eldest and youngest member of the club? **1**

d) Find an expression for the constant term in the expansion of $\dfrac{1}{x^2}\left(x^5 - \dfrac{3}{x}\right)^{16}$. **3**

e) Newton's law of cooling states that the rate of cooling of a body is proportional to the excess of the temperature of a body above the surrounding temperature.
The temperature of a bowl of soup satisfies an equation of the form $T = B + Ae^{-kt}$ where T is the temperature of the soup, t is time in minutes, A and k are constants, and B is the temperature of the surroundings.
The bowl of soup cools from $96°C$ to $88°C$ in 3 minutes in a room of temperature $25°C$.

i) Find the exact values of A and k. **2**

ii) Find the temperature of the bowl of soup, to the nearest degree, after a further 10 minutes have passed. **1**

End of Questions 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) The acceleration of a particle moving in a straight line is given by

$$\frac{d^2x}{dt^2} = -\frac{98}{x^2}$$

when x metres is the displacement from the origin after t seconds. When $t = 0$, the particle is 4 metres to the right of the origin with a velocity of 7 metres per second.

You may use the result $\frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

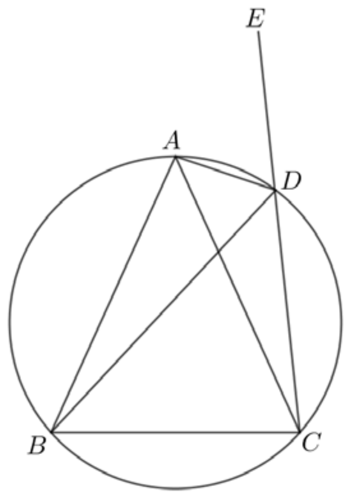
i) Show that the velocity, v, of the particle, in terms of x, is $v = \frac{14}{\sqrt{x}}$. **3**

ii) Find an expression for t in terms of x. **2**

iii) How many seconds does it take for the particle to reach a point 121 metres to the right of the origin? **1**

iv) Find the displacement of the particle after 15 seconds. Round your answer to the nearest metre. **1**

b) $ABCD$ is a cyclic quadrilateral in which $AB = AC$, and CD is produced to E.



Not to Scale

Copy or trace the diagram into your writing booklet.

i) Explain why $\angle ABC = \angle ADE$. **1**

ii) Hence or otherwise, prove that AD bisects $\angle BDE$. **2**

Question 13 continues on page 10

Question 13 (continued)

c) i) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)\,.$$

ii) Use this result to show that $2^2 + 4^2 + 6^2 + \cdots + 100^2 = 171700$. **1**

iii) Hence evaluate $1^2 + 3^2 + 5^2 + \cdots + 99^2$. **1**

End of Questions 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) A man standing 100 metres from the base of a high-rise building observes an external lift moving up the outside wall of the building at a constant rate of 6.5 metres per second.

i) If θ radians is the angle of elevation of the lift from the observer, show that $\dfrac{d\theta}{dt} = \dfrac{13\cos^2\theta}{200}$. **2**

ii) Evaluate $\dfrac{d\theta}{dt}$ at the instant when the lift is 40 metres above the observer's horizontal line of vision. **2**

b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. $S(0, a)$ is the focus of the parabola. The normal to the parabola at P meets the y-axis at R.

i) Given that the equation of the normal at P is $x + py = ap^3 + 2ap$ (DO NOT PROVE THIS). Find the coordinates of R. **1**

ii) N lies on PR so that $SN \perp PR$. Find the coordinates of N. **2**

iii) Show that, as P moves along the parabola, the locus of N is another parabola. Find the vertex and focus of this parabola. **2**

c) A projectile is fired from the origin O with velocity V and with angle of elevation θ, where $\theta \neq \dfrac{\pi}{2}$. You may assume that

$$x = Vt\cos\theta \quad \text{and} \quad y = -\frac{1}{2}gt^2 + Vt\sin\theta,$$

where x and y are the horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing.

i) Show that the equation of flight of the projectile can be written as **2**

$$y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta), \quad \text{where } h = \frac{V^2}{2g}.$$

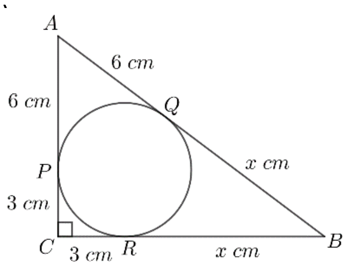
ii) Show that the point (X, Y), where $X \neq 0$, can be hit by firing at two different angles θ_1 and θ_2 provided $X^2 < 4h(h - Y)$. **2**

iii) Show that no point above the x-axis can be hit by firing at two different angles θ_1 and θ_2, satisfying $\theta_1 < \dfrac{\pi}{4}$ and $\theta_2 < \dfrac{\pi}{4}$. **2**

End of Paper

2018 Mathematics Extension 1 Trial Solutions

Section 1

1	**B** $\int \sin^2 8x\,dx$ $= \int \frac{1}{2}(1 - \cos 16x)dx$ $= \frac{1}{2}\left(x - \frac{1}{16}\sin 16x\right) + C$ $= \frac{1}{2}x - \frac{1}{32}\sin 16x + C$	1 Mark
2	**C** $f(x) = \frac{5}{x+2} + 3$ Asymptotes of $f(x)$ are $x = -2$ and $y = 3$ Asymptotes of $f^{-1}(x)$ are $x = 3$ and $y = -2$	1 Mark
3	**C** $y = \frac{1}{2}\cos^{-1} x - \frac{\pi}{2}$ Range for: $y = \cos^{-1} x \rightarrow\ 0 \le y \le \pi$ $y = \frac{1}{2}\cos^{-1} x \rightarrow\ 0 \le y \le \frac{\pi}{2}$ $y = \frac{1}{2}\cos^{-1} x - \frac{\pi}{2} \rightarrow\ -\frac{\pi}{2} \le y \le 0$	1 Mark
4	**A** 4 vowels, A, I, U, E 4! ways of arranging all the vowels 1 group of vowels and 5 other letters $6! \times 4!$	1 Mark
5	**B** Let the ratio be $k{:}\,1$ $8 = \frac{5k + (-4) \times 1}{k+1}$ $8k + 8 = 5k - 4$ $3k = -12$ $k = -4$ $\therefore -4 : 1$	1 Mark
6	**B** $x = 6\sin 5t + 8\cos 5t$ $\dot{x} = 30\cos 5t - 40\sin 5t$ Amplitude (greatest speed) $= \sqrt{30^2 + 40^2} = 50$	1 Mark
7	**C** Let $BR = x\ cm$ $CP = CR = 3\ cm;\ AP = AQ = 6\ cm; BQ = BR = x\ cm$ (two tangents from an external point have equal lengths) $AC^2 + BC^2 = AB^2$ (Pythagoras' Theorem\` $(6+3)^2 + (3+x)^2 = (6+x)^2$ $81 + 9 + 6x + x^2 = 36 + 12x + x^2$ $6x = 54$ $x = 9$ $BR = 9\ cm$ $BC = 3 + 9 = 12\ cm$  	1 Mark

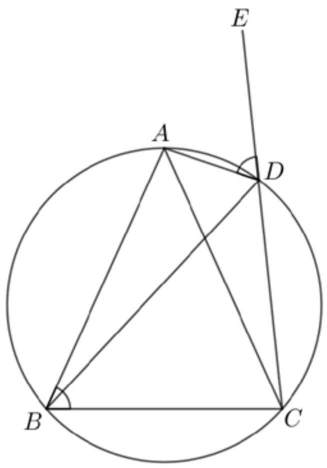
8	**D** $\frac{^nP_r}{^nC_r} = \frac{n!}{(n-r)!} \div \frac{n!}{(n-r)!\,r!}$ $\frac{^nP_r}{^nC_r} = \frac{n!}{(n-r)!} \times \frac{(n-r)!\,r!}{n!}$ $\frac{^nP_r}{^nC_r} = r!$	1 Mark
9	**A** $\cos 2x = p$ $2x = 2n\pi \pm \cos^{-1} p$ $x = n\pi \pm \frac{1}{2}\cos^{-1} p$	1 Mark
10	**D** When $t = 0, x = 7$ Only B or D are possible. In D, $x = 3 + 4e^t$ $v = \frac{dx}{dt} = 4e^t$ $4e^t = x - 3$ $v = x - 3$	1 Mark

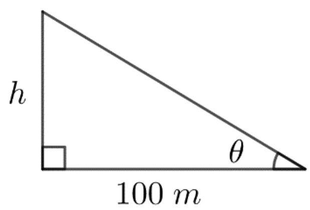
Section 2		
Q11 a)	$\dfrac{4x}{x+5} < 2 \qquad x \neq -5$ $4x(x+5) < 2(x+5)^2$ $4x(x+5) - 2(x+5)^2 < 0$ $(x+5)\big(4x - 2(x+5)\big) < 0$ $(x+5)(2x-10) < 0$ $(x+5)(x-5) < 0$ $-5 < x < 5$	2 Marks Correct solution 1 Mark Correctly identifies 5 and -5 as important, or equivalent merit
Q11 b)	$x - 5y = 0$ $y = \dfrac{1}{5}x$ $m_1 = \dfrac{1}{5}$ $2x + y - 3 = 0$ $y = -2x + 3$ $m_2 = -2$ $\tan\theta = \left\lvert \dfrac{\frac{1}{5} - (-2)}{1 + \frac{1}{5} \times (-2)} \right\rvert$ $\tan\theta = \dfrac{11}{3}$ $\theta = 74°44'41.57''$ $\theta = 74°45'$ (nearest minute)	2 Marks Correct solution 1 Mark Correctly identifies the gradients of both lines and attempts to substitute into the formula
Q11 c)	$\dfrac{d}{dx}(x\sin^{-1}x + \sin^{-1}x)$ $= \dfrac{d}{dx}\big((x+1)\sin^{-1}x\big)$ $= \sin^{-1}x + \dfrac{x+1}{\sqrt{1-x^2}}$	2 Marks Correct solution 1 Mark Differentiate $\sin^{-1}x$ correctly
Q11 d)	$x = 5\cos\theta$ $\cos\theta = \dfrac{x}{5}$ $y = 5\sin\theta$ $\sin\theta = \dfrac{y}{5}$ $\sin^2\theta + \cos^2\theta = 1$ $\left(\dfrac{y}{5}\right)^2 + \left(\dfrac{x}{5}\right)^2 = 1$ $\dfrac{y^2}{25} + \dfrac{x^2}{25} = 1$ $x^2 + y^2 = 25$	2 Marks Correct solution 1 Mark Makes significant progress to eliminate θ
Q11 e) i)	Let $t = \tan\frac{\theta}{2}$ $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{\frac{2t}{1+t^2}}{1 - \frac{1-t^2}{1+t^2}}$ $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{2t}{1+t^2-(1-t^2)}$ $\dfrac{\sin\theta}{1-\cos\theta} = \dfrac{2t}{2t^2}$ $\dfrac{\sin\theta}{1-\cos\theta} == \dfrac{1}{t}$	2 Marks Correct solution 1 Mark Correct substitution of t formula

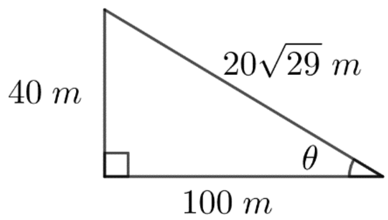
	$\dfrac{\sin\theta}{1-\cos\theta}=\dfrac{1}{\tan\left(\frac{\theta}{2}\right)}$ $\dfrac{\sin\theta}{1-\cos\theta}=\cot\left(\frac{\theta}{2}\right)$	
Q11 e) ii)	$\dfrac{\pi}{2}<\theta<\pi$ $\sin\theta=\dfrac{5}{6}$ $\cos\theta=-\dfrac{\sqrt{11}}{6}$ $\dfrac{\pi}{4}<\dfrac{\theta}{2}<\dfrac{\pi}{2}$ $\cot\left(\frac{\theta}{2}\right)=\dfrac{\sin\theta}{1-\cos\theta}$ $\cot\left(\frac{\theta}{2}\right)=\dfrac{\frac{5}{6}}{1-\left(-\frac{\sqrt{11}}{6}\right)}$ $\cot\left(\frac{\theta}{2}\right)=\dfrac{\frac{5}{6}}{\frac{6+\sqrt{11}}{6}}$ $\cot\left(\frac{\theta}{2}\right)=\dfrac{5}{6+\sqrt{11}}$	2 Marks Correct solution 1 Mark Find the correct exact value of $\cos\theta$
Q11 f)	$I=\displaystyle\int_{\ln\frac{1}{2}}^{\ln\frac{\sqrt{3}}{2}}\frac{e^x}{1+4e^{2x}}dx$ Let $u=2e^x$ $du=2e^x dx$ $x=\ln\dfrac{\sqrt{3}}{2}, u=\sqrt{3}$ $x=\ln\dfrac{1}{2}, u=1$ $I=\dfrac{1}{2}\displaystyle\int_{\ln\frac{1}{2}}^{\ln\frac{\sqrt{3}}{2}}\frac{2e^x}{1+(2e^x)^2}dx$ $I=\dfrac{1}{2}\displaystyle\int_{1}^{\sqrt{3}}\frac{du}{1+u^2}$ $I=\dfrac{1}{2}[\tan^{-1}(u)]_1^{\sqrt{3}}$ $I=\dfrac{1}{2}\left[\tan^{-1}\sqrt{3}-\tan^{-1}1\right]$ $I=\dfrac{1}{2}\left(\dfrac{\pi}{3}-\dfrac{\pi}{4}\right)$ $I=\dfrac{\pi}{24}$	3 Marks Correct solution 2 Marks Correct primitive function 1 Mark Obtains $du=2e^x dx$ and correct boundary values in terms of u rather than x
Q12 a) i)	$f(x)=\log_e(2x-1)+x^3+1$ $f(0.6)=\log_e(2\times0.6-1)+0.6^3+1=-0.3934\ldots$ $f(0.6)<0$ $f(0.7)=\log_e(2\times0.7-1)+0.7^3+1=0.4267\ldots$ $f(0.7)>0$ Since there is a sign change, and the function is continuous for $0.6\le x\le0.7$, therefore there exist a root between 0.6 and 0.7.	1 Mark Correction solution

Q12 a) ii)	$f(x) = \log_e(2x-1) + x^3 + 1$ $f(0.6) = \log_e(2 \times 0.6 - 1) + 0.6^3 + 1$ $f'(x) = \dfrac{2}{2x-1} + 3x^2$ $f'(0.6) = \dfrac{2}{2 \times 0.6 - 1} + 3 \times 0.6^2$ $x = 0.6 - \dfrac{f(0.6)}{f'(0.6)}$ $x = 0.6 - \dfrac{\log_e(2 \times 0.6 - 1) + 0.6^3 + 1}{\dfrac{2}{2 \times 0.6 - 1} + 3 \times 0.6^2}$ $x = 0.6355 \ldots$ $x = 0.636$ (3 significant figures)	2 Marks Correct solution 1 Mark Correct differentiation of $f(x)$ and substitution of 0.6
Q12 b) i)	$3x^3 - 7x - 2 = 0$ $\alpha + \beta + \gamma = 0$ $\alpha\beta + \alpha\gamma + \beta\gamma = -\dfrac{7}{3}$ $\alpha\beta\gamma = \dfrac{2}{3}$ $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = \dfrac{-\frac{7}{3}}{\frac{2}{3}}$ $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = -\dfrac{7}{2}$	2 Marks Correct solution 1 Mark Obtains correct sum and product of roots
Q12 b) ii)	$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $\alpha^2 + \beta^2 + \gamma^2 = 0 - 2 \times -\dfrac{7}{3}$ $\alpha^2 + \beta^2 + \gamma^2 = \dfrac{14}{3}$	2 Marks Correct solution 1 Mark Correct manipulation of the algebraic expression
Q12 c) i)	Choosing 2 men from 11 men and 2 women from 9 women ${}^{11}C_2 \times {}^{9}C_2 = 1980$	1 Mark Correct solution
Q12 c) ii)	2 member has already been set, so only choosing 2 people from the left over 18 people. ${}^{18}C_2 = 153$	1 Mark Correct solution
Q12 d)	$\left(x^5 - \dfrac{3}{x}\right)^{16} = \displaystyle\sum_{k=0}^{16} {}^{16}C_k (x^5)^{16-k}(-3x^{-1})^k$ $\left(x^5 - \dfrac{3}{x}\right)^{16} = \displaystyle\sum_{k=0}^{16} {}^{16}C_k x^{80-5k}(-3)^k x^{-k}$ $\left(x^5 - \dfrac{3}{x}\right)^{16} = \displaystyle\sum_{k=0}^{16} {}^{16}C_k (-3)^k x^{80-6k}$ Constant term in the expansion of: $\dfrac{1}{x^2}\left(x^5 - \dfrac{3}{x}\right)^{16} \rightarrow x^{-2} \times x^{80-6k} = x^0$	3 Marks Correct solution 2 Marks Attempts to solve k by matching up the correct terms in the expansion 1 Mark Obtains the correct expression for the binomial expansion

	$-6k = -78$ $k = 13$ Constant term is ${}^{16}C_{13}(-3)^{13}$	
Q12 e) i)	$T = B + Ae^{-kt}$ $t = 0, B = 25, T = 96$ $96 = 25 + Ae^0$ $A = 71$ $t = 3, B = 25, T = 88$ $88 = 25 + 71e^{-k\times3}$ $\frac{63}{71} = e^{-3k}$ $\ln\frac{63}{71} = \ln e^{-3k}$ $-3k = \ln\frac{63}{71}$ $k = -\frac{1}{3}\ln\frac{63}{71}$	2 Marks Correct solution 1 Mark Obtains the correct value for A or k
Q12 e) ii)	Further 10 minutes $t = 3 + 10 = 13$ $T = 25 + 71e^{-\left(-\frac{1}{3}\ln\frac{63}{71}\right)\times13}$ $T = 67.29\ldots$ $T = 67°C$ (nearest degree)	1 Mark Correct solution
Q13 a) i)	$\frac{d^2x}{dt^2} = -\frac{98}{x^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ $\frac{1}{2}v^2 = \int -98x^{-2}dx$ $\frac{1}{2}v^2 = \frac{-98x^{-1}}{-1} + C$ $v^2 = \frac{196}{x} + C$ $x = 4, v = 7$ $7^2 = \frac{196}{4} + C$ $C = 0$ $v^2 = \frac{196}{x}$ $\therefore v = \frac{14}{\sqrt{x}}\ \ (v > 0)$	3 Marks Correct solution 2 Marks Correct integral for $\frac{1}{2}v^2$ and attempt to find C 1 Mark Express $\frac{1}{2}v^2 = \int -98x^{-2}dx$
Q13 a) ii)	$v = \frac{dx}{dt} = \frac{14}{\sqrt{x}}$ $\frac{dx}{dt} = \frac{14}{x^{\frac{1}{2}}}$ $\frac{dt}{dx} = \frac{x^{\frac{1}{2}}}{14}$ $t = \frac{1}{14}\int x^{\frac{1}{2}}dx$ $t = \frac{1}{14}\times\frac{x^{\frac{3}{2}}}{\frac{3}{2}}dx$ $t = \frac{1}{21}x^{\frac{3}{2}} + C$	2 Marks Correct solution 1 Mark Correct expression for t and attempt to find C

	$t = 0, x = 4$ $0 = \frac{1}{21} \times 4^{\frac{3}{2}} + C$ $C = -\frac{8}{21}$ $\therefore t = \frac{1}{21}x^{\frac{3}{2}} - \frac{8}{21}$	
Q13 a) iii)	$x = 121$ $t = \frac{1}{21} \times 121^{\frac{3}{2}} - \frac{8}{21}$ $t = 63$ seconds $\therefore$ It took 63 seconds for the particle to reach a point 121 metres to the right of the origin.	1 Mark Correct solution
Q13 a) iv)	$t = 15$ $15 = \frac{1}{21}x^{\frac{3}{2}} - \frac{8}{21}$ $15 + \frac{8}{21} = \frac{1}{21}x^{\frac{3}{2}}$ $x^{\frac{3}{2}} = \frac{323}{21} \div \frac{1}{21}$ $x^{\frac{3}{2}} = 323$ $x = 323^{\frac{2}{3}}$ $x = 47.07 \ldots$ $x = 47\ m$ (nearest metre) $\therefore$ The particle is at 47 metres to the right of the origin after 15 seconds.	1 Mark Correct solution
Q13 b) i)	$\angle ABC = \angle ADE$ (exterior angle of a cyclic quadrilateral is equal to its opposite interior angle) 	1 Mark Correct solution
Q13 b) ii)	Let $\angle ABC = \theta$ ΔABC is an isosceles triangle ($AB = AC$) $\angle ABC = \angle ACB = \theta$ (Equal base angles of isosceles ΔABC) $\angle ACB = \angle ADB = \theta$ (Angles in the same segment) $\angle ABC = \angle ADE$ (shown in the previous part) $\angle ADB = \angle ADE = \theta$ $\therefore AD$ bisects $\angle BDE$.	2 Marks Correct solution 1 Mark Identify $\angle ACB = \angle ADB$ and provided correct reasoning

Q13 c) i)	$1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$ 1. Prove statement is true for $n=1$. $LHS=1^2$ $LHS=1$ $RHS=\frac{1}{6}\times 1\times(1+1)\times(2\times 1+1)$ $RHS=\frac{1}{6}\times 2\times 3$ $RHS=1$ $LHS=RHS$ ∴ Statement is true for $n=1$ 2. Assume statement is true for $n=k$ (k some positive integer) $i.e. 1^2+2^2+3^2+\cdots+k^2=\frac{1}{6}k(k+1)(2k+1)$ 3. Prove statement is true for $n=k+1$ $i.e. 1^2+2^2+3^2+\cdots+k^2+(k+1)^2=\frac{1}{6}(k+1)(k+2)(2k+3)$ $LHS=1^2+2^2+3^2+\cdots+k^2+(k+1)^2$ $LHS=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$ $LHS=\frac{1}{6}(k+1)[k(2k+1)+6(k+1)]$ $LHS=\frac{1}{6}(k+1)[2k^2+k+6k+6]$ $LHS=\frac{1}{6}(k+1)[2k^2+7k+6]$ $LHS=\frac{1}{6}(k+1)(k+2)(2k+3)$ $LHS=RHS$ ∴ Statement is true by mathematical induction for all integers $n\geq 1$.	3 Marks Correct solution 2 Marks Makes significant progress in proving the statement involving $n=k+1$ 1 Mark Establishes result for $n=1$
Q13 c) ii)	$2^2+4^2+6^2+\cdots+100^2$ $=2^2(1^2+2^2+3^2+\cdots+50^2)$ $=4\times\frac{1}{6}\times 50\times(50+1)(2\times 50+1)$ $=171700$	1 Mark Correct solution
Q13 c) iii)	$1^2+3^2+5^2+\cdots+99^2$ $=(1^2+2^2+3^2+\cdots.+99^2+100^2)-(2^2+4^2+6^2+\cdots+100^2)$ $=\frac{1}{6}\times 100\times(100+1)\times(2\times 100+1)-171700$ $=166650$	1 Mark Correct solution
Q14 a) i)	$\tan\theta=\frac{h}{100}$ $\sec^2\theta\,d\theta=\frac{1}{100}dh$ $\frac{d\theta}{dh}=\frac{1}{100\sec^2\theta}$ $\frac{d\theta}{dh}=\frac{\cos^2\theta}{100}$ 	2 Marks Correct solution 1 Mark Obtains the correct expression of $\frac{d\theta}{dh}$ in terms of θ

	Or $\theta = \tan^{-1}\frac{h}{100}$ $\frac{d\theta}{dh} = \frac{100}{100^2 + h^2}$ $\frac{d\theta}{dh} = \frac{100}{100^2 + 100^2\tan^2\theta}$ $\frac{d\theta}{dh} = \frac{100}{100^2(1 + \tan^2\theta)}$ $\frac{d\theta}{dh} = \frac{1}{100\sec^2\theta}$ $\frac{d\theta}{dh} = \frac{\cos^2\theta}{100}$ $\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt}$ $\frac{d\theta}{dt} = \frac{\cos^2\theta}{100} \times 6.5$ $\frac{d\theta}{dt} = \frac{13\cos^2\theta}{200}$	
Q14 a) ii)	When $h = 40$ $\tan\theta = \frac{40}{100}$ $\sqrt{40^2 + 100^2}$ $= \sqrt{11600}$ $= 20\sqrt{29}$  $\cos\theta = \frac{100}{20\sqrt{29}}$ $\cos^2\theta = \frac{10000}{11600}$ $\cos^2\theta = \frac{25}{29}$ $\frac{d\theta}{dt} = \frac{13\cos^2\theta}{200}$ $\frac{d\theta}{dt} = \frac{13}{200} \times \frac{25}{29}$ $\frac{d\theta}{dt} = \frac{13}{232}\ rad/s$	2 Marks Correct solution 1 Mark Obtains the correct value for $\cos^2\theta$
Q14 b) i)	At $R, x = 0$ $x + py = ap^3 + 2ap$ $py = ap^3 + 2ap$ $y = ap^2 + 2a$ $\therefore R(0, ap^2 + 2a)$	1 Mark Correct solution
Q14 b) ii)	$m_{SN} = m_T$ (gradient of the tangent at P) $x^2 = 4ay$ $y = \frac{x^2}{4a}$ $\frac{dy}{dx} = \frac{2x}{4a}$ $\frac{dy}{dx} = \frac{x}{2a}$ $m_T = \frac{2ap}{2a}$ $m_T = p$	2 Marks Correct solution 1 Mark Obtains the correct equation of SN

	Equation of SN $y - a = p(x - 0)$ $y = px + a$ Sub into $x + py = ap^3 + 2ap$ to find N $x + p(px + a) = ap^3 + 2ap$ $x + p^2x + ap = ap^3 + 2ap$ $x(1 + p^2) = ap^3 + ap$ $x = \frac{ap(p^2+1)}{p^2+1}$ $x = ap$ $y = p \times ap + a$ $y = ap^2 + a$ $\therefore N(ap, ap^2 + a)$	
Q14 b) iii)	$N(ap, ap^2 + a)$ $x = ap$ $p = \frac{x}{a}$ $y = ap^2 + a$ $y = a \times \left(\frac{x}{a}\right)^2 + a$ $y = \frac{x^2}{a} + a$ $ay = x^2 + a^2$ $x^2 = ay - a^2$ $x^2 = a(y - a)$ $\therefore$ Locus of N is another parabola with vertex $(0, a)$, the focal length is $\frac{a}{4}$, focus $\left(0, \frac{5a}{4}\right)$	2 Marks Correct solution 1 Mark Obtains the locus of N
Q14 c) i)	$x = Vt\cos\theta \ldots\ldots (1)$ $y = -\frac{1}{2}gt^2 + Vt\sin\theta \ldots\ldots (2)$ From (1) $t = \frac{x}{V\cos\theta}$ Substitute into (2) $y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + V\left(\frac{x}{V\cos\theta}\right)\sin\theta$ $y = -\frac{gx^2}{2V^2\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$ $y = -\frac{gx^2\sec^2\theta}{2V^2} + x\tan\theta$ $y = -\frac{2gx^2\sec^2\theta}{4V^2} + x\tan\theta$ $y = \frac{2g}{V^2} \times -\frac{x^2\sec^2\theta}{4} + x\tan\theta$ $y = \frac{1}{h} \times -\frac{x^2\sec^2\theta}{4} + x\tan\theta \qquad \left(\frac{V^2}{2g} = h\right)$ $y = -\frac{x^2(1+\tan^2\theta)}{4h} + x\tan\theta \qquad (\sec^2\theta = 1 + \tan^2\theta)$ $y = x\tan\theta - \frac{1}{4h}x^2(1 + \tan^2\theta)$	2 Marks Correct solution 1 Mark Substitute $t = \frac{x}{V\cos\theta}$ into y and attempts to simplify

Q14 c) ii)	$y = x\tan\theta - \frac{1}{4h}x^2(1+\tan^2\theta)$ $4hy = 4hx\tan\theta - x^2(1+\tan^2\theta)$ $4hy = 4hx\tan\theta - x^2 - x^2\tan^2\theta$ $x^2\tan^2\theta - 4hx\tan\theta + x^2 + 4hy = 0$ Substitute (X, Y) into the above: $X^2\tan^2\theta - 4hX\tan\theta + X^2 + 4hY = 0$ This quadratic equation in $\tan\theta$ has two distinct roots if $\Delta > 0$. $(-4hX)^2 - 4X^2(X^2 + 4hY) > 0$ $16h^2X^2 - 4X^4 - 16hX^2Y > 0$ $4h^2 - X^2 - 4hY > 0$ (since $X^2 > 0$) $4h^2 - 4hY > X^2$ $X^2 < 4h^2 - 4hY$ $X^2 < 4h(h - Y)$ If $X^2 < 4h(h-Y)$, there are two solutions, $\tan\theta_1$ and $\tan\theta_2$, for the equation, two differeng angles, θ_1 and θ_2 can be used to hit the point (X, Y).	2 Marks Correct solution 1 Mark Substitute (X, Y) and forms quadratic equation in $\tan\theta$
Q14 c) iii)	Let $\tan\theta_1, \tan\theta_2$ be roots of $X^2\tan^2\theta - 4hX\tan\theta + X^2 + 4hY = 0$ Product of roots $\tan\theta_1\tan\theta_2 = \frac{X^2 + 4hY}{X^2}$ $\tan\theta_1\tan\theta_2 = 1 + \frac{4hY}{X^2}$ $\tan\theta_1\tan\theta_2 > 1$ $(X^2 > 0, Y > 0)$ If both $0 < \theta_1 < \frac{\pi}{4}$ and $0 < \theta_2 < \frac{\pi}{4}$, then $0 < \tan\theta_1 < 1$ and $0 < \tan\theta_2 < 1$, so $\tan\theta_1\tan\theta_2 < 1$. This contradicts to the product of roots. $\therefore$ No point above the x-axis can be hit from two different angles θ_1 and θ_2 satisfying $\theta_1 < \frac{\pi}{4}$ and $\theta_2 < \frac{\pi}{4}$.	2 Marks Correct solution 1 Mark Obtains the expression $\tan\theta_1\tan\theta_2 > 1$ from the product of roots